Relativistic calculation of the triton binding energy and its implications

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First results for the triton binding energy obtained from the relativistic spectator or Gross equation are reported. The Dirac structure of the nucleons is taken into account. Numerical results are presented for a family of realistic OBE models with off-shell scalar couplings. It is shown that these off-shell couplings improve both the fits to the two-body data and the predictions for the binding energy.

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The first realistic nonrelativistic calculations of the triton binding energy were completed in the 1970's [1]. Later it was shown that different methods gave the same results, and that the binding energy could be calculated to an accuracy of a few keV by considering all nucleon-nucleon (NN) partial waves up to $j = 4$ [2]. Today, if three-body forces (3BFs) are not considered, a small discrepancy of about 0.5-1.0 MeV remains between the experimentally observed value of -8.48 MeV and values obtained from realistic nonrelativistic NN potentials. State-of-the art calculations now include sophisticated 3BFs, and when their strength is adjusted to give the correct triton binding energy, an excellent value is also obtained for the ^{4}He binding energy (and to a lesser extent other light nuclei up to $A \simeq 7$) [3].

However, relativistic effects should make a contribution to the binding energy at the level of several hundred keV. Using a mean momentum of about 200 MeV (consistent with nonrelativistic estimates) we expect to see corrections of the order of $(v/c)^2 \simeq (p/m)^2 \simeq 4\%$. If this is 4% of the binding energy, then it amounts to about 300 keV. However, if relativity has a greater effect on the attractive σ exchange part of the force (as it does in nuclear matter calculations using the Walecka model) then we might obtain an effect 10 times larger.

The importance of this problem has been recognized, and relativistic effects have been estimated using a separable kernel in the Bethe-Salpeter equation [4], assuming minimal relativity in the Blankenbecler-Sugar equation [5], and by adding corrections of first order in $(v/c)^2$ to the Schrödinger equation [6]. All of these calculations include some contributions coming from relativistic kinematics, but none treats the full Dirac structure of the nucleons, or investigates effects which might arise from a realistic relativistic treatment of the *NN dynamics*.

The purpose of this letter is to present the first numerical calculations of the triton binding energy obtained from the manifestly covariant three-body spectator (or Gross) equations for three identical spin 1/2 particles, and to discuss the implications of these calculations. Some preliminary results were reported in conference proceedings [7].

The three-body spectator equations were first introduced and applied to scalar particles in 1982 [8], and then extended to the case of three spin 1/2 particles in lectures given at the University of Hannover soon afterward. Recently, a more tractable form for the equations has been developed, and a full derivation of the equations will be published elsewhere [9]. In this letter we describe only a few of their features briefly.

In the absence of 3BFs the three-body scattering amplitude is obtained from a sum of all successive two-body scatterings. Because the three particles are identical, each two-body scattering differs from the others only by a permutation, and they can therefore all be summed by one operator equation of the form

$$|\Gamma^1\rangle = 2M^1 G^1 \mathcal{P}_{12} |\Gamma^1\rangle \,, \tag{1}$$

where $|\Gamma^1\rangle$ is a vertex function describing the contribution to the bound state from all processes in which the 23 pair was the last to interact (with particle 1 a spectator), the two-body amplitude M^1 describes the scattering of the 23 pair, G^1 is the propagator for the 23 pair, and P_{12} is a permutation operator interchanging particles 1 and 2. (The factor of 2 comes from the contribution of P_{13} which equals the one of P_{12}.)

The three-body spectator equations have the same structure as (1), but incorporate the additional feature that the spectator is restricted to its positive energy mass-shell in all intermediate states. With the conventions implied above, consistency also requires that particle 2 be on-shell, so that two particles are always on-shell. We think of these constraints as a reorganization of Eq. (1) which will, in some cases, improve its convergence. The constraints are manifestly covariant, and lead to the following equation

$$|\Gamma^1_2\rangle = 2M^1_{22} G^1_2 \mathcal{P}_{12} |\Gamma^1_2\rangle \,, \tag{2}$$

where the lower index labels the second on-shell particle. Hence only particle 3, the (unique) particle which has just left one interaction and is about to enter another one, is off-shell in Eq. (2).

To reduce Eq. (2) to a practical form, we take matrix elements of the operators using three-particle helicity states similar to those defined by Wick [10]. Both ρ-spin states (where $\rho = +$ is the u spinor positive energy state and $\rho = -$ is the v spinor negative energy state) of the off-shell particle must be treated. The three-body states will be written in the abbreviated form $|J1(23)\rho\rangle$, where J is the total angular momentum of the state, ρ the ρ-spin of the off shell particle, $1 = \{q, \lambda_1\}$ (where q and λ_1 are the magnitude of the three momentum and the helicity of the spectator in the three-body c.m.), and $(23) = \{\tilde{p}, j, m_j, \lambda_2, \lambda_3\}$ (where $\tilde{p}$ is the magnitude of the relative three momentum of the 23 system, j and m_j are the angular momentum of the pair and its projection in the direction of $\mathbf{q}$, and λ_2 and λ_3 are the helicities of particles 2 and 3, *all defined in the rest frame of the 23 pair*). We will suppress all isospin indices. Using this notation, the final form of the three-body spectator equation for Γ^1 is

$$\begin{aligned}\langle J1(23)\rho|\Gamma^1\rangle = &\sum_{j'm'} \sum_{\substack{\lambda''_2\lambda''_3\,\rho'' \\ \lambda'_1\lambda'_2\lambda'_3\,\rho'}} \int_0^{q_{\rm crit}} q'^2 dq' \frac{m}{E_{q'}} \int_0^\pi d\chi \sin\chi \\ &\times \langle j(23)\rho|M^1|j(2''3'')\rho''\rangle \, \frac{m}{E_{\tilde{p}''}} \, g^{\rho''}(q,\tilde{p}'') \\ &\times \mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')] \, \frac{m}{E_{\tilde{p}'}} \, \langle J'1'(2'3')\rho'|\Gamma^1\rangle \,, \end{aligned} \tag{3}$$

where $\mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')]$ is the matrix element of the permutation operator, given below, and $g^\rho(q,\tilde{p})$ the propagator of the off-shell particle in different ρ-spin states

$$g^{+}(q,\tilde{p}) = \frac{1}{2E_{\tilde{p}} - W_q}\,, \qquad g^{-}(q,\tilde{p}) = -\frac{1}{W_q}\,. \tag{4}$$

Because four-momentum is conserved in the relativistic formalism, the mass W_q of the 23 pair depends on q,

$$W_q^2 = M_t^2 + m^2 - 2M_t E_q\,, \tag{5}$$

with $E_q = \sqrt{m^2+\mathbf{q}^2}$. Note that Eq. (3) includes a sum over intermediate helicities and angular momentum quantum numbers, and an integration over the internal spectator momentum q' and the angle χ between the directions of $\mathbf{q}'$ and $\mathbf{q}$. The integration over q' has been limited to the finite interval $[0, q_{\rm crit}]$, where $q_{\rm crit}$ is the root of the equation $W_{q_{\rm crit}} = 0$. At this critical spectator momentum (equal to $\simeq 4m/3 \simeq 1200$ MeV), the two-body subsystem is recoiling at the speed of light and the relativistic effects are enormous! Contributions for $q' > q_{\rm crit}$ are very small, and come from two-body states with *spacelike* four-momenta. It seems sensible to simply neglect the region $q' \geq q_{\rm crit}$ and set the three-body amplitudes to zero there. As it turns out, the solutions go smoothly to zero as $q \to q_{\rm crit}$ anyway, so we may impose the condition that they are zero beyond this point without making the amplitudes discontinuous in q.

Exchanging particles 1 and 2 implies that particle 2 becomes the spectator and now its momentum and helicity must be expressed in the c.m. frame of the three-body system, while the variables of particles 1 and 3 must be expressed in the rest frame of the 13 pair. Boosting from one frame to another introduces Wigner rotations of both the single particle and two-body helicities. The final result for the permutation operator is

$$\begin{aligned}\mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')] &= (-1)^{m-\lambda_1+\lambda_3'}\sqrt{2j+1}\sqrt{2j'+1}\\ &\times d^{(J)}_{m-\lambda_1,m'-\lambda_2'}(\chi)\, d^{(j)}_{m,\lambda_2''-\lambda_3''}(\tilde{\theta}'')\, d^{(j')}_{m',\lambda_1'-\lambda_3'}(\tilde{\theta}')\\ &\times d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1)\, d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2)\, \mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'}(q,q',\chi)\,,\end{aligned} \tag{6}$$

where the functions $d^{(1/2)}_{m_1,m_2}(\beta)$ are the Wigner rotation matrices, and $\mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'}(q,q',\chi)$ describes *exactly* the Wigner rotations of the off-shell particle 3, as well as the nontrivial matrix elements between the different ρ-spinors u and v of particle 3 as they appear in the rest frames of the 23 pair and the 13 pair.

We have solved Eq. (3) numerically for a variety of realistic NN models. The two-body amplitudes obtained for all of these models result from an exact solution of the two-body spectator equation, as described in Ref. [11], and are therefore fully consistent with the three-body equations.

These models will be described in detail elsewhere. Briefly, they are all one-boson exchange (OBE) models with a kernel composed of the exchange of 6 commonly used bosons: the $\pi, \eta, \sigma, \delta, \omega$, and ρ. The parameters of each model were determined by fitting to the NN phase shifts below 350 MeV and to deuteron properties.

In all cases the following pion coupling was used:

$$\begin{aligned}g_\pi\Lambda_\pi &= g_\pi\left[\gamma^5 - \frac{\nu_\pi}{2m}\left[(m-\not{p}')\,\gamma^5 + \gamma^5\,(m-\not{p})\right]\right]\\ &= g_\pi\left[(1-\nu_\pi)\gamma^5 + \frac{\nu_\pi}{2m}\gamma^5\,\not{q}\right]\,,\end{aligned} \tag{7}$$

where p and p' are the four momenta of the incoming and outgoing nucleons, and the couplings proportional to ν_π do not contribute if the nucleons are on-shell. In this family, we fixed $g_\pi^2/4\pi = 13.34$ and chose $\nu_\pi = 1$, giving the conventional pseudovector pion coupling with large off-shell effects.

A particular feature of these models, and a central point of this letter, is that they *also* include phenomenological scalar σ (with $I=0$) and δ ($I=1$) exchanges with off-shell scalar-nucleon-nucleon (sNN) couplings of the form

$$\begin{aligned}g_s\Lambda_s(p',p) = g_s\Big[1 &- \frac{\nu_s}{2m}\,(m-\not{p}' + m - \not{p})\\ &+ \frac{\kappa_s}{4m^2}\,(m-\not{p}')\,(m-\not{p})\Big]\,.\end{aligned} \tag{8}$$

The vertex $\Lambda_s(p',p)$ given in Eq. (8) is the *most general form* the sNN vertex can take, but as far as we know the off-shell scalar couplings which depend on ν_s and κ_s have never been studied previously. The family of models discussed here has $\kappa_s = 0$ and values of ν varying from $0 \to 2.6$, where

$$\nu_\sigma = -0.75\,\nu \qquad \nu_\delta = 2.60\,\nu\,. \tag{9}$$

We will see that these couplings proportional to ν are extremely important.

The results of our calculations are summarized in Fig. 1. The lower panel of Fig. 1 shows how χ^2 for the fits to the two-body data varies with ν for this family of OBE models. Fits were done for values of $\nu =$ 0, 0.5, 1.0, 1.6, 1.8, 1.9, 2.0, 2.2, and 2.6, and the dashed curve smoothly interpolates these individual cases. We emphasize that *each of these models with different values of* ν *are realistic* in the sense that for each case OBE parameters (13 in all) were adjusted to give the best possible fit to the NN data below 350 MeV. Although the 13 parameters differ only slightly from case to case, the models are not quite equivalent. The figure shows that there is a significant variation in the quality of the fit; the best models lie in the region $1.5 \leq \nu \leq 2.0$ (all with $\chi^2 \leq 2.45$ as shown in the figure). This probably rules out the model with $\nu = 0$ (for example). We conclude that the introduction of these ν-dependent couplings significantly improves the fit to the two-body data and that the implicit choice of $\nu = 0$ made in previous work is not optimal.

The upper panel of Fig. 1 shows the variation of the three-body binding energy with ν. The rapid dependence of the binding energy on ν is rather striking. An increase in ν from 0 to 1.6 changes our prediction from -6.24 to

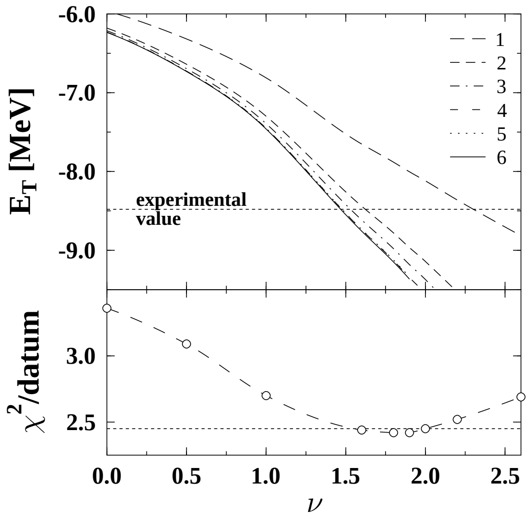

FIG. 1. Triton binding energy E_t for specific values of j_{max} (upper panel) and χ^2 for the fits to the two-body data (lower panel) versus the scalar meson off-shell parameter ν defined in the text. The curves in both panels are smooth interpolations through the actual calculations. The lower panel also includes the line $\chi^2 = 2.45$ for reference.

-8.76 MeV, and a value in good agreement with experiment would be obtained for $\nu \simeq 1.5$, still in the range of ν's which give the best fit to the two-body data.

The panel and Table 1 (for the cases with smaller ν) also show how the binding energy converges as the number of three-body partial waves, characterized by the highest included pair angular momentum j_{max}, increases. Because of the large increase in the predicted values as the number of channels increases from 28 ($j_{max} = 1$) to 52 ($j_{max} = 2$), we were concerned about the convergence of the three-body calculations and studied it in detail. We carried the calculations all the way to $j_{max} = 6$ with 148 channels. We find that the individual contributions from channels with odd j tend to cancel while those from channels with even j are all attractive. Thus, the steps from even to odd j_{max} are small compared to those from odd to even j_{max}. From a detailed study of the individual contributions we estimate that the results are converged to about 1 keV for $\nu = 0$ and to about 5 keV for $\nu = 1.6$.

We conclude that the best of the two-body models examined so far yield a three-body binding energy from about -8.5 to -9.5 MeV. In subsequent work we will display the dependence of these results on the boosts, the negative ρ-spin states, and other relativisitc effects, and we will study additional two-body models. Here we will discuss the origin and implications of the ν dependence which we have observed.

To understand why the binding energy is so sensitive to ν, we may look at the half off-shell Born amplitude for scalar exchange (i.e. the amplitude with *one* nucleon off-shell). For the positive ρ-spin sector, we have

TABLE I. Absolute values of the triton binding energies in MeV. The first row is the result when only $^1S_0$ and $^3S_1$-$^3D_1$ positive energy channels are included. The other rows show results obtained when all channels with two-body angular momentum $j \leq j_{max}$ are included. The total number of three-body channels in each case is N.

j_{max}	N	coupling parameter ν			
		0.0	0.5	1.0	1.6
1^+	5	6.003	6.345	6.850	7.769
1	28	5.963	6.318	6.812	7.652
2	52	6.180	6.639	7.299	8.441
3	76	6.214	6.695	7.393	8.615
4	100	6.232	6.726	7.452	8.740
5	124	6.233	6.726	7.452	8.736
6	148	6.235	6.731	7.461	8.757

$$\mathcal{V}_s = \frac{g_s^2 \left\{\overline{u}(\mathbf{p}') \left[1 - \frac{\nu_s}{2m}\,(m - \not{p})\right] u(\mathbf{p})\right\} \; \{\overline{u}(-\mathbf{p}')u(-\mathbf{p})\}}{m_s^2 - (p'-p)^2}$$
$$\simeq V_s \left[1 - \nu_s \frac{2E_p - W}{2m}\right] = C_s V_s \,, \qquad (10)$$

where V_s is the usual scalar potential obtained from such a reduction when $\nu_s = 0$, $p = (W - E_p, \mathbf{p})$ is the momentum of the off-shell particle, W is the energy of the two-body system in its rest frame, and we have ignored the lower components of the Dirac spinors in carrying out the reduction. The effect of the ν_s dependence is to multiply the scalar potential by the factor $C_s = [1 - \nu_s(2E_p - W)/2m]$. In applications to two-body scattering, the ν-dependent term is a small correction with a sign depending on the energy, but in the three-body bound state it is always positive. Assuming an average nucleon momentum of about 200 MeV gives roughly a 10% variation over the range of ν from 0 to 2. The observed variation of about 4 MeV over this range is explained therefore if the average strength of the σ-exchange potential is about 40 MeV, which is the right order of magnitude. This shows how the large variation in binding energy which we observe can be explained by a "small" relativistic effect.

An OBE model with off-shell couplings has a very rich structure. For example, consider two successive interactions of a scalar meson with a single nucleon. The vertex function (8) contains the operator $m - \not{p}$ which is just the inverse of the nucleon propagator, so that it can remove the nucleon propagator and contract the two interaction vertices to a single vertex describing the emission of two mesons from a single point. If the two mesons emerging from this point are coupled to a second nucleon they generate a triangle or a bubble diagram. These diagrams are two-boson exchange terms similar to those (involving pions) which would emerge from a nonlinear sigma model. Alternatively, if these two mesons couple to two *different* nucleons, they generate diagrams usually associated with three-body forces. It is easy to generalize this result: *an*

OBE model with off-shell couplings is equivalent to another OBE model without these couplings, but with an additional specific family of N-boson exchange diagrams and N-body forces.

We conclude with two observations. The discovery that off-shell scalar couplings play an important role in both improving the description of two-body data and in predicting three-body binding energies would only have been possible in the context of a relativistic formalism closely connected to an effective field theory. The fact that these couplings are equivalent to a strong energy dependence in the context of nonrelativistic theory is precisely the reason they could not have been discovered there; nonrelativistic potentials are supposed to be energy independent. In the context of an effective field theory, however, these are a natural and legitimate extension of the simplest assumption about the spin structure of the sNN vertex. The most general sNN vertex was given in Eq. (8) above, and can have only three different spin couplings. Once the third term depending on κ_s in (8) is studied, all of the possibilities will have been exhausted. In this way an effective field theory is tightly constrained, even if some of its interactions are strongly energy dependent in a nonrelativistic context.

We believe that this way of looking at dynamics may very well be the most significant contribution to come from relativistic methods. The traditional arguments suggesting that relativistic effects are very small refer to relativistic *kinematics* only. As Eq. (8) illustrates, relativistic *dynamics* provides a new way to study nuclei, even at low energies.

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